

ORBITS OF FINITE SOLVABLE GROUPS ON CHARACTERS

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ABSTRACT. We prove that if a solvable group A acts coprimely on a solvable group G , then A has a “large” orbit in its corresponding action on the set of ordinary complex irreducible characters of G . This extends (at the cost of a weaker bound) a 2005 result of A. Moretó who obtained such a bound in case that A is a p -group.

1. INTRODUCTION

The main purpose of this paper is to generalize a 2005 result of A. Moretó on orbits in a certain group action. While there are many results on the orbits in the action of a group acting via automorphisms on some other group (of which the action of linear groups on their natural modules is a particularly prominent example), Moretó’s result is noteworthy in that it is one of the very few dealing with the action of a group on the set of irreducible characters of another group.

More precisely, let the finite group A act (via automorphisms) on the finite group G . Such an action induces an action of A on the set $\text{Irr}(G)$ in an obvious way (where $\text{Irr}(G)$ denotes the set of complex irreducible characters of G). When G is elementary abelian, we are back to studying linear group actions and all the known results apply. But for nonabelian G not much is known about this interesting action. Note that when $(|A|, |G|) = 1$, then it is well-known that the orbit sizes of A on $\text{Irr}(G)$ are the same as the orbit sizes in the natural action of A on the conjugacy classes of G , and this latter action was of some importance in [5], where some specialized results on this action were obtained. But apart from these we are aware only of two major results on the action of A on $\text{Irr}(G)$.

The first such result is due to D. Gluck [2]. He proved that when A is abelian and G is solvable, then there always exists an “arithmetically large” orbit on $\text{Irr}(G)$ (i.e., an orbit whose size is divisible by “many” different primes).

The second result is the 2005 result [10] by Moretó mentioned above. It proves the existence of a “large” orbit on $\text{Irr}(G)$ in case that A is a p -group for some prime p and G is solvable such that $(|A|, |G|) = 1$.

In this paper we take the second result to the next level and establish the existence of a large orbit on $\text{Irr}(G)$ in case that A is solvable and G is solvable such that $(|A|, |G|) = 1$. More precisely, our main result is the following.

2000 *Mathematics Subject Classification.* 20C20.

Part of this work was done while the first author was on sabbatical leave and was visiting the University of Wisconsin-Parkside. He thanks the Mathematics Department there for its hospitality.

Theorem A. *Let A and G be finite solvable groups such that A acts faithfully and coprimely on G . Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Then $|A| \leq b^{49}$.*

We make a few observations. First, it has already been observed in [10] that the coprimeness assumption cannot be omitted.

Second, not surprisingly, our bound is weaker than the bound obtained in [10]. While in [10], when A is a p -group, the existence of an orbit of size roughly $|A|^{\frac{1}{19}}$ on $\text{Irr}(G)$ is proved, in the more general situation of Theorem A we only get an orbit of size about $|A|^{\frac{1}{49}}$. Both bounds, however, are far from best possible anyway and thus the results are more of a qualitative nature. The true bound is probably close to $|A|^{\frac{1}{2}}$ in both cases.

Third, if we assume $|AG|$ to contain no small primes, the bounds one can get tend to be much better. In [10] it is observed in passing that if odd order is assumed, then the bound will get much better. And here we will explicitly establish a better bound in case that $|G|$ is not divisible by 6 (see Theorem 3.3 below).

Our proof of Theorem A is largely based on the ideas introduced in [10] and extends them to the more general hypothesis of Theorem A. In particular, our proof does not use Moretó's result, but rather reproves it (with a weaker bound) as a special case. To keep the bound from getting too large, we also make use of a recent strengthening in the solvable case of a result by M. Aschbacher and R. Guralnick [1] on the size of $|G/G'|$ of a linear solvable group; see 2.2.

2. THE ABELIAN QUOTIENT OF LINEAR GROUPS

We first recall a result due to Aschbacher and Guralnick.

Proposition 2.1. *Let G be a finite solvable group that acts faithfully and completely reducibly on a finite vector space V . Then $|G : G'| \leq |V|$.*

Proof. This is a special case of the much more general [1, Theorem 3]. □

Next we provide a recent strengthening of Proposition 2.1 by the authors.

Theorem 2.2. *Let G be a finite solvable group that acts faithfully and completely reducibly on a finite vector space V , and let B be the size of the largest orbit of G on V . Then $|G : G'| \leq B$.*

Proof. This will appear in [6]. □

3. MAIN THEOREM

The following result is an extension of [10, Lemma 2.1].

Theorem 3.1. *Assume that a solvable π -group A acts faithfully on a solvable π' -group G . Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Let $\Gamma = AG$ be the semidirect product. Let $K_{i+1} = \mathbf{F}_{i+1}(\Gamma)/\mathbf{F}_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall π -subgroup of K_{i+1} for all $i \geq 1$. Let $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma)) = V_{i1} + V_{i2}$ where V_{i1} is the π part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$*

and V_{i2} is the π' part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ for all $i \geq 1$. Let $K \triangleleft \Gamma$ such that $\mathbf{F}_i(\Gamma) \triangleleft K$. Let $L_{i+1,\pi} = K_{i+1,\pi} \cap K$. We have that $|\mathbf{C}_{L_{i+1,\pi}}(V_{i1})| \leq b^2$, and $|\mathbf{C}_{L_{i+1,\pi}}(V_{i1})| \leq b$ if $L_{i+1,\pi}$ is abelian. The order of the maximum abelian quotient of $\mathbf{C}_{L_{i+1,\pi}}(V_{i1})$ is less than or equal to b for all $i \geq 1$.

Proof. We know that $L_{i+1,\pi}$ acts faithfully and completely reducibly on $V_{i1} + V_{i2}$. Clearly $\mathbf{C}_{L_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i2} and $L_{i+1,\pi}/\mathbf{C}_{L_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i1} .

Let L be the pre-image of $\mathbf{C}_{L_{i+1,\pi}}(V_{i1})$ in $(\Gamma/\mathbf{F}_{i-1}(\Gamma)/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma)))/V_{i1}$. Write $L = QV_{i2}$, where $Q \in \text{Hall}_\pi(L)$. We have to prove that $|Q| \leq b^2$. Clearly $\mathbf{F}(L) = V_{i2}$ and $\Phi(L) = 1$. We know by a theorem of Brauer that Q acts faithfully on $\text{Irr}(V_{i2})$. Replacing A by a conjugate, if necessary, we may assume that $Q \leq A$. It follows from our hypothesis that $|Q : \mathbf{C}_Q(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$.

Now, let $\lambda \in \text{Irr}(V_{i2})$. By [3, Theorem 13.28], there exists $\chi \in \text{Irr}(G)$ lying over λ that is $\mathbf{C}_Q(\lambda)$ -invariant. We claim that $\mathbf{C}_Q(\chi) = \mathbf{C}_Q(\lambda)$. It is clear that $|Q : \mathbf{C}_Q(\lambda)|$ divides the degree of any character of L lying over λ . Therefore, $|Q : \mathbf{C}_Q(\lambda)|$ divides the degree of any character of QG lying over λ since the pre-image of L in Γ is normal in Γ . Now, [3, Corollary 8.16] and Clifford's correspondence [3, Theorem 6.11] yield that there exist $\psi \in \text{Irr}(QG)$ lying over χ , whence over λ , such that $\psi(1)_\pi = |Q : \mathbf{C}_Q(\chi)|$. It follows that $\mathbf{C}_Q(\chi) = \mathbf{C}_Q(\lambda)$, as desired. In particular, $|Q : \mathbf{C}_Q(\lambda)| \leq b$. We deduce that for all $\lambda \in \text{Irr}(V_{i2})$, $|Q : \mathbf{C}_Q(\lambda)| \leq b$. Now, [4], for instance, implies that $|Q| \leq b^2$. Also, if Q is abelian, then $|Q| \leq b$. The order of the maximum abelian quotient of Q is less than or equal to b by Theorem 2.2. This completes the proof of the theorem. \square

Now we are ready to prove Theorem A, which we restate.

Theorem 3.2. *Let A be a solvable π -group that acts faithfully on a solvable π' -group G . Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Then $|A| \leq b^{49}$.*

Proof. Let $\Gamma = AG$ be the semidirect product of A and G . By Gaschutz's theorem, $\Gamma/\mathbf{F}(\Gamma)$ acts faithfully and completely reducibly on $\text{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$. It follows from [8, Theorem 3.3] that there exists $\lambda \in \text{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$ such that $T = \mathbf{C}_\Gamma(\lambda) \leq \mathbf{F}_8(\Gamma)$.

Let $K_{i+1} = \mathbf{F}_{i+1}(\Gamma)/\mathbf{F}_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall π -subgroup of K_{i+1} for all $i \geq 1$.

We know that $K_{i+1,\pi}$ acts faithfully and completely reducibly on $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$. It is clear that we may write $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma)) = V_{i1} + V_{i2}$ where V_{i1} is the π part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ and V_{i2} is the π' part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ for all $i \geq 1$. Clearly $\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i2} . Thus $|\mathbf{C}_{K_{i+1,\pi}}(V_{i1})| \leq b^2$ and the order of the maximum abelian quotient of $\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ is less than or equal to b by Theorem 3.1. Also $K_{i+1,\pi}/\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i1} . Since $K_{i+1,\pi}/\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ is nilpotent, $|K_{i+1,\pi}/\mathbf{C}_{K_{i+1,\pi}}(V_{i1})| \leq |V_{i1}|^\beta/2$ where $\beta = \log(32)/\log(9)$ by [7, Theorem 3.3]. Also the order of the maximum abelian quotient of $K_{i+1,\pi}/\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ is bounded above by $|V_{i1}|$ by Proposition 2.1.

Thus we have the following, $|K_{2,\pi}| \leq b^2$ and the order of the maximum abelian quotient of $K_{2,\pi}$ is bounded above by b .

$|K_{3,\pi}| \leq |\mathbf{C}_{K_{3,\pi}}(V_{21})| \cdot |K_{3,\pi}/\mathbf{C}_{K_{3,\pi}}(V_{21})| \leq b^2 \cdot b^\beta$ and the order of the maximum abelian quotient of $K_{3,\pi}$ is bounded above by $b \cdot b = b^2$.

$|K_{4,\pi}| \leq |\mathbf{C}_{K_{4,\pi}}(V_{31})| \cdot |K_{4,\pi}/\mathbf{C}_{K_{4,\pi}}(V_{31})| \leq b^2 \cdot b^{2\beta}$ and the order of the maximum abelian quotient of $K_{4,\pi}$ is bounded above by $b \cdot b^2 = b^3$.

$|K_{5,\pi}| \leq |\mathbf{C}_{K_{5,\pi}}(V_{41})| \cdot |K_{5,\pi}/\mathbf{C}_{K_{5,\pi}}(V_{41})| \leq b^2 \cdot b^{3\beta}$ and the order of the maximum abelian quotient of $K_{5,\pi}$ is bounded above by $b \cdot b^3 = b^4$.

$|K_{6,\pi}| \leq |\mathbf{C}_{K_{6,\pi}}(V_{51})| \cdot |K_{6,\pi}/\mathbf{C}_{K_{6,\pi}}(V_{51})| \leq b^2 \cdot b^{4\beta}$ and the order of the maximum abelian quotient of $K_{6,\pi}$ is bounded above by $b \cdot b^4 = b^5$.

$|K_{7,\pi}| \leq |\mathbf{C}_{K_{7,\pi}}(V_{61})| \cdot |K_{7,\pi}/\mathbf{C}_{K_{7,\pi}}(V_{61})| \leq b^2 \cdot b^{5\beta}$ and the order of the maximum abelian quotient of $K_{7,\pi}$ is bounded above by $b \cdot b^5 = b^6$.

$|K_{8,\pi}| \leq |\mathbf{C}_{K_{8,\pi}}(V_{71})| \cdot |K_{8,\pi}/\mathbf{C}_{K_{8,\pi}}(V_{71})| \leq b^2 \cdot b^{6\beta}$.

Next, we show that $|\Gamma : T|_\pi \leq b$.

Let χ be any irreducible character of G lying over λ . Then every irreducible character of Γ that lies over χ also lies over λ and hence has degree divisible by $|\Gamma : T|$. But χ extends to its stabilizer in Γ and thus some irreducible character of Γ lying over χ has degree $\chi(1)|A : C_A(\chi)|$. The π -part of $|\Gamma : T|$, therefore, divides $|A : \mathbf{C}_A(\chi)|$, which is at most b .

This gives that $|A| \leq b^2 \cdot b^2 \cdot b^\beta \cdot b^2 \cdot b^{2\beta} \cdot b^2 \cdot b^{3\beta} \cdot b^2 \cdot b^{4\beta} \cdot b^2 \cdot b^{5\beta} \cdot b^2 \cdot b^{6\beta} \cdot b = b^{15+21\beta} \leq b^{48.124}$ \square

Theorem 3.3. *Let A be a solvable π -group that acts faithfully on a solvable π' -group G . Assume that $2, 3 \notin \pi$. Let b be an integer such that $|A : \mathbf{C}_A(\chi)| \leq b$ for all $\chi \in \text{Irr}(G)$. Then $|A| \leq b^4$.*

Proof. Let $\Gamma = AG$ be the semidirect product of A and G . By Gaschutz's theorem, $\Gamma/\mathbf{F}(\Gamma)$ acts faithfully and completely reducibly on $\text{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$. It follows from [9, Theorem 3.2] that there exists $\lambda \in \text{Irr}(\mathbf{F}(\Gamma)/\Phi(\Gamma))$ and $K \triangleleft \Gamma$ such that $T = \mathbf{C}_\Gamma(\lambda) \subseteq K$, $\mathbf{F}(\Gamma) \subseteq K \subseteq \mathbf{F}_3(\Gamma)$. The π -subgroup of $K\mathbf{F}_2(\Gamma)/\mathbf{F}_2(\Gamma)$ and the π -subgroup of $(K \cap \mathbf{F}_2(\Gamma))/\mathbf{F}(\Gamma)$ are abelian.

Let $K_{i+1} = \mathbf{F}_{i+1}(\Gamma)/\mathbf{F}_i(\Gamma)$ and let $K_{i+1,\pi}$ be the Hall π -subgroup of K_{i+1} for all $i \geq 1$.

We know that $K_{i+1,\pi}$ acts faithfully and completely reducibly on $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$. It is clear that we may write $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma)) = V_{i1} + V_{i2}$ where V_{i1} is the π part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ and V_{i2} is the π' part of $K_i/\Phi(\Gamma/\mathbf{F}_{i-1}(\Gamma))$ for all $i \geq 1$. Clearly $\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i2} and $K_{i+1,\pi}/\mathbf{C}_{K_{i+1,\pi}}(V_{i1})$ acts faithfully and completely reducibly on V_{i1} .

Since the image of $K \cap \mathbf{F}_2(\Gamma)$ in $K_{2,\pi}$ is abelian, $|(K \cap \mathbf{F}_2(\Gamma))/\mathbf{F}(\Gamma)| \leq b$ and the order of the maximum abelian quotient of $K_{2,\pi}$ is bounded above by b by Theorem 3.1.

Since $L_{3,\pi} = K/\mathbf{F}_2(\Gamma)$ is abelian, $|L_{3,\pi}| \leq |\mathbf{C}_{L_{3,\pi}}(V_{21})| \cdot |L_{3,\pi}/\mathbf{C}_{L_{3,\pi}}(V_{21})| \leq b \cdot b$ by Theorem 3.1 and Proposition 2.1.

Next, we show that $|\Gamma : T|_\pi \leq b$.

Let χ be any irreducible character of G lying over λ . Then every irreducible character of Γ that lies over χ also lies over λ and hence has degree divisible by $|\Gamma : T|$. But χ extends to its stabilizer in Γ and thus some irreducible character of Γ lying over χ has degree $\chi(1)|A : C_A(\chi)|$. The π -part of $|\Gamma : T|$, therefore, divides $|A : \mathbf{C}_A(\chi)|$, which is at most b .

This gives that $|A| \leq b \cdot b \cdot b \cdot b \leq b^4$. \square

Since when $(|A|, |G|) = 1$, the orbit sizes of A on $\text{Irr}(G)$ are the same as the orbit sizes in the natural action of A on the conjugacy classes of G , the following results immediately follow from the previous ones.

Theorem 3.4. *Let A be a solvable π -group that acts faithfully on a solvable π' -group G . Let b be an integer such that $|A : \mathbf{C}_A(C)| \leq b$ for all $C \in \text{cl}(G)$. Then $|A| \leq b^{49}$.*

Theorem 3.5. *Let A be a solvable π -group that acts faithfully on a solvable π' -group G . Assume that $2, 3 \notin \pi$. Let b be an integer such that $|A : \mathbf{C}_A(C)| \leq b$ for all $C \in \text{cl}(G)$. Then $|A| \leq b^4$.*

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